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A NEW COMPUTER FOR ALGEBRAIC FUNCTIONS OF A COMPLEX VARIABLE

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ABSTRACT

A NEW COMPUTER
FOR
ALGEBRAIC FUNCTIONS OF A COMPLEX VARIABLE

The ESIAC algebraic computer is presented as a new tool for use in control system analysis and design. The ESIAC is a specialized analog computer which serves the designer as a "two-dimensional slide rule" for functions of a complex variable and their pole-zero plots.

The features of the ESIAC are described and a number of applications discussed, including its use in automatic plotting of root locus and frequency response plots, in the evaluation of residues, in factoring polynomials, in network synthesis, in the analysis and design of carrier systems, systems with dead time and sampled data systems, in the treatment of non-linear systems by the describing-function method, and in the design of systems by power-density spectra methods.

A NEW COMPUTER FOR ALGEBRAIC FUNCTIONS OF A COMPLEX VARIABLE

by Merle L. Morgan

Introduction

The problems associated with the development of modern guidance systems, process controls, and other servomechanisms have placed increasingly greater demands on the design engineer. The complex nature of these problems has resulted in the development of more effective methods of analysis and synthesis for their solution. For example, in the last decade there has been rapid development and extensive use of Laplace transform techniques. The representation of functions by pole-zero diagrams and root locus plots in the study of the transient and steady-state responses of systems has become widespread.

The advantage of using Laplace transform techniques is that complicated integro-differential equations in the time domain are converted to simple algebraic equations in the frequency domain. This allows the engineer to do his mathematical manipulation in terms of algebraic functions rather than differential equations. These algebraic techniques are applicable not only to ordinary linear servomechanism design and network synthesis but also to the analysis and design of carrier systems, systems with dead time, sampled data systems, and various nonlinear systems by the describing function method, and to the design of systems by power density spectra methods.

Evaluation of Algebraic Functions

The algebraic functions of a complex variable and associated pole-zero plots encountered in such problems are commonly evaluated graphically. As the systems become more complicated, however, the graphical techniques become quite time consuming and the use of a computer is very desirable.

Unfortunately, the design engineer has not previously had at his disposal a simple computer for algebraic functions described in terms of their zeros and poles. A large digital computer can be programmed to handle such functions but the use of its equipment and personnel is costly, and its operating time usually must be scheduled in advance. For these reasons, the designer cannot quickly obtain a solution for one trial set of parameters and then proceed immediately with a solution for a new set chosen upon inspection of the first solution.

Analog methods are typically much simpler and more suitable for calculations where the high accuracy of digital methods is not required. Conventional analog computers, however, are designed to solve differential equations involving functions of the real variable time and are not easily adapted to functions of a complex variable.

Two-dimensional electric potential distribution methods on the other hand have a very simple, direct relationship to functions of a complex variable. Potential distribution analogs have been used very effectively as an aid in visualizing functions in a complex plane. They offer the additional advantage that any conformal transformation of the coordinate system can be made without affecting the function, providing proper boundary values are maintained. Their practical application to computation, however, has been limited in the past to the direct potential analog of a function, in which current sources and sinks are placed at the zeros and poles of the function. This provides a direct analog of the logarithm of the magnitude of the function, but no similar analog can be constructed in general for the angle of the function.

The new ESIAC[*] Algebraic Computer[o] provides for the first time analog computation of both the magnitude and the angle of a function of a complex variable. This is accomplished by means of a combination of electric potential distribution and lumped circuit elements. With the ESIAC the advantages of analog computation are available for evaluation of the functions of a complex variable encountered in all the algebraic system design methods previously mentioned. In transient analysis problems, the ESIAC provides an improvement on the graphical root locus technique. The ESIAC also measures the residues at the first order poles of a function, thus providing the partial fraction expansion for the transient response of a system. For frequency response calculations, the frequency axis of the complex plane is scanned to plot the gain and phase shift of various portions of a system as a function of frequency.

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[*] Trade mark of Electro-Measurements, Inc., Portland, Oregon, U. S. A.

[o] U. S. A. and foreign patents pending

Factored Algebraic Functions

Algebraic functions of a complex variable are sometimes expressed as the ratio of two polynomials. When the polynomials are factored, however, such a function can always be expressed in either of the forms:

$$F = \pm K_1 s^{n_0} (s - s_1)^{n_1} (s - s_2)^{n_2} (s - s_3)^{n_3} \ldots$$

$$= \pm K_1 s^{n_0} \prod_i (s - s_i)^{n_i} \qquad (1)$$

or

$$F = \pm K_2 s^{n_0} \left(1 - \frac{s}{s_1}\right)^{n_1} \left(1 - \frac{s}{s_2}\right)^{n_2} \left(1 - \frac{s}{s_3}\right)^{n_3} \ldots$$

$$= \pm K_2 s^{n_0} \prod_i \left(1 - \frac{s}{s_i}\right)^{n_i} \qquad (2)$$

in which

F, s and the s_i's represent complex vector quantities.
K_1 or K_2, n_0 and the n_i's represent scalar quantities which can be either positive or negative.

Any rational algebraic function can be factored into either of the above forms, and the Laplace transform of any linear differential equation can therefore be expressed in this form. The ESIAC handles functions expressed in the form of either of these two equations.

Such factored functions are evaluated by recognizing that the magnitude of F is equal to the product of the magnitudes of the factors on the right hand side of the equation (its log magnitude is equal to the sum of the log magnitudes on the right) and the angle of F is equal to the sum of the angles of the factors on the right hand side of the equation.

In practical engineering problems, it is usually necessary to determine the values of $|F|$ (magnitude of F) and $\angle F$ (angle of F) corresponding to each of a large number of values of s. Sometimes the values are used directly as in plots of gain and phase as a function of frequency. In other cases, a large number of trial values of s must be substituted in the equation to find the values of s which satisfy the equation when the value of F is specified. These values are the roots of the equation. This technique is used in the analysis of control systems by the root locus method. The same approach is also applicable to the factoring of high order algebraic polynomials, an operation which is often required as a preliminary to the application of the root locus method to the equation of a system.

The numerical and graphical methods for solving such problems are tedious since the magnitude and angle of $(s - s_i)$ or $(1 - \frac{s}{s_i})$ must be determined for each factor for every trial value of s. The unique advantage of the ESIAC is that a change in s involves setting only the dials for the magnitude and angle of s. The values of the s_i's and other parameters are established at the start of the problem and require no further attention as the values of $|F|$ and $\angle F$ are determined at various values of s.

Description of the ESIAC

The ESIAC consists of a console with a plotting board at the center and panels containing the principal controls conveniently arranged around it , as shown in Figure 1. A close-up of the operating controls is shown in Figure 2, and in Figure 3 the covers at each side of the plotting board have been raised to show the two electric potential analog sheets and the frames for positioning and moving probes over them. The supply of probes and probe clamps in the drawer is also shown in Figure 3.

The ESIAC contains two separate analog circuits--one for $|F|$ and the other for $\angle F$. In setting up a problem, a pattern of probes is placed in the frame over each sheet at positions corresponding to the s_i's (the zeros and poles) in the equation. The frames are then moved over the sheets to positions corresponding to values of the complex variable s. The voltage distributions on the sheets are such that the voltage at each probe represents on one sheet the magnitude and on the other sheet the angle of one factor $(s - s_i)$ or $(1 - \frac{s}{s_i})$. The exponent n_i on each factor is established by selecting the value of an electric coupling element in the clamp which attaches the probe to the frame. Dials on the panel provide for setting the values of F, K and s, and for selecting

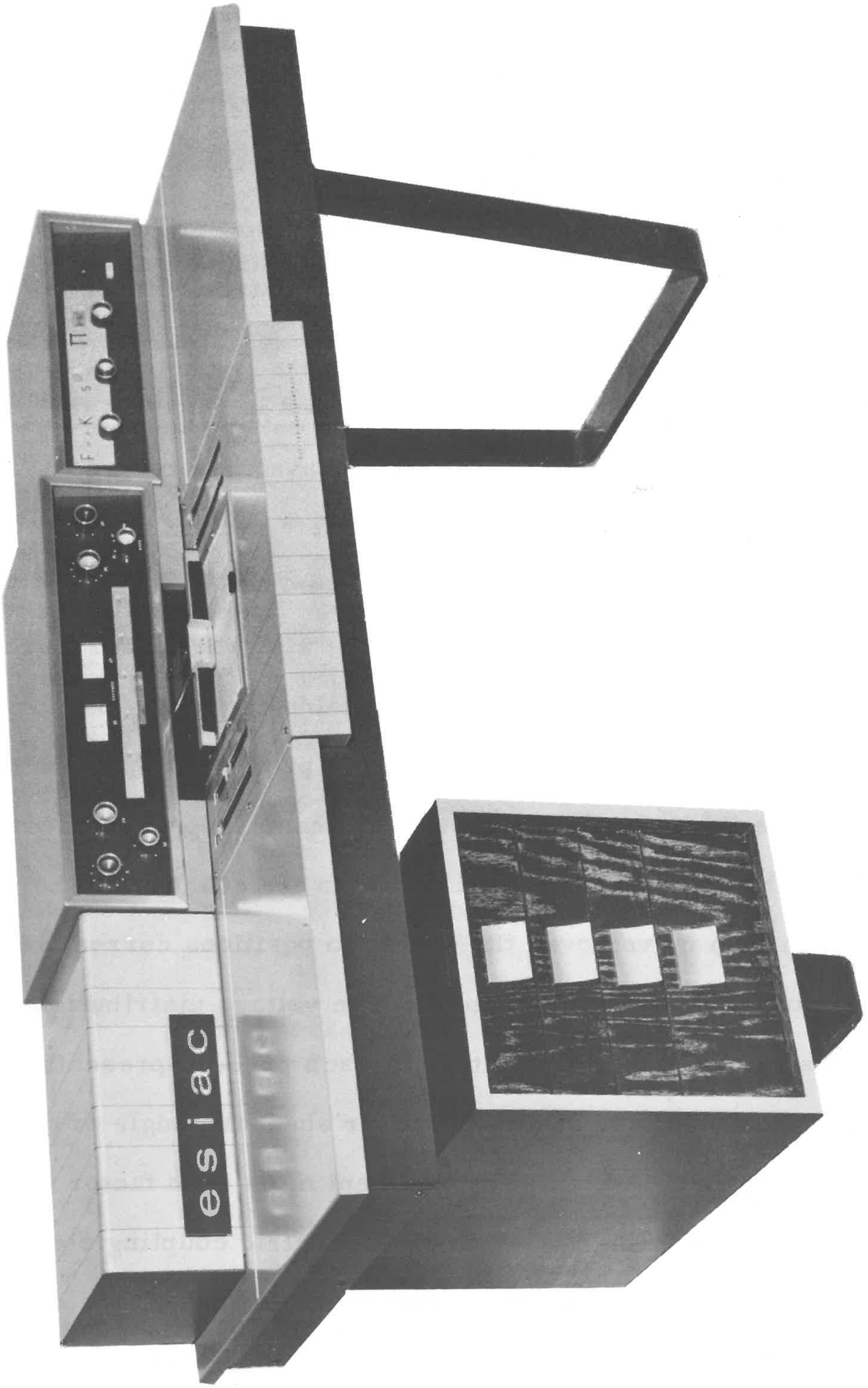


FIGURE 1. THE ESIAC

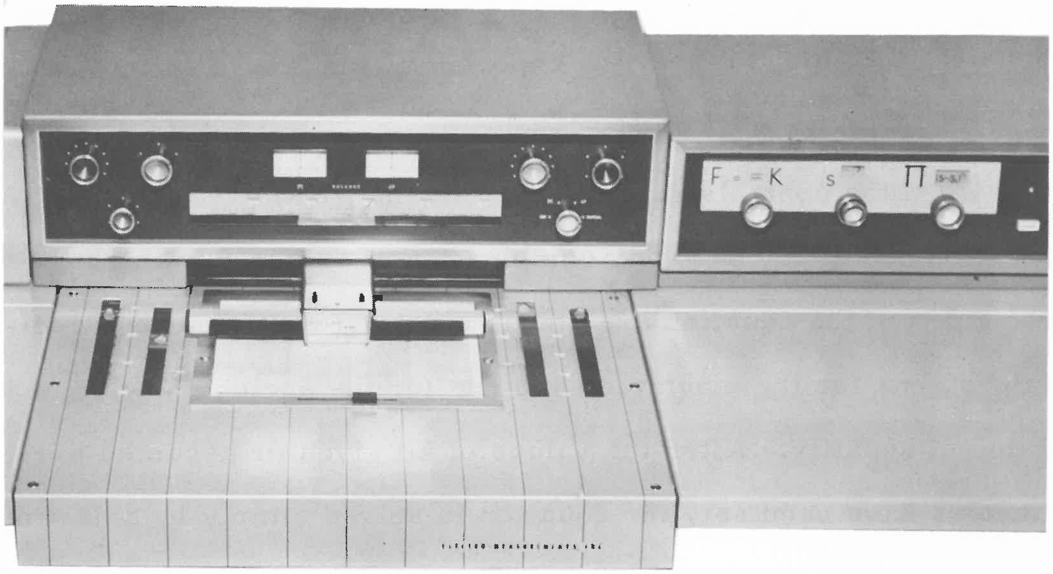


FIGURE 2. ESIAC OPERATING CONTROLS

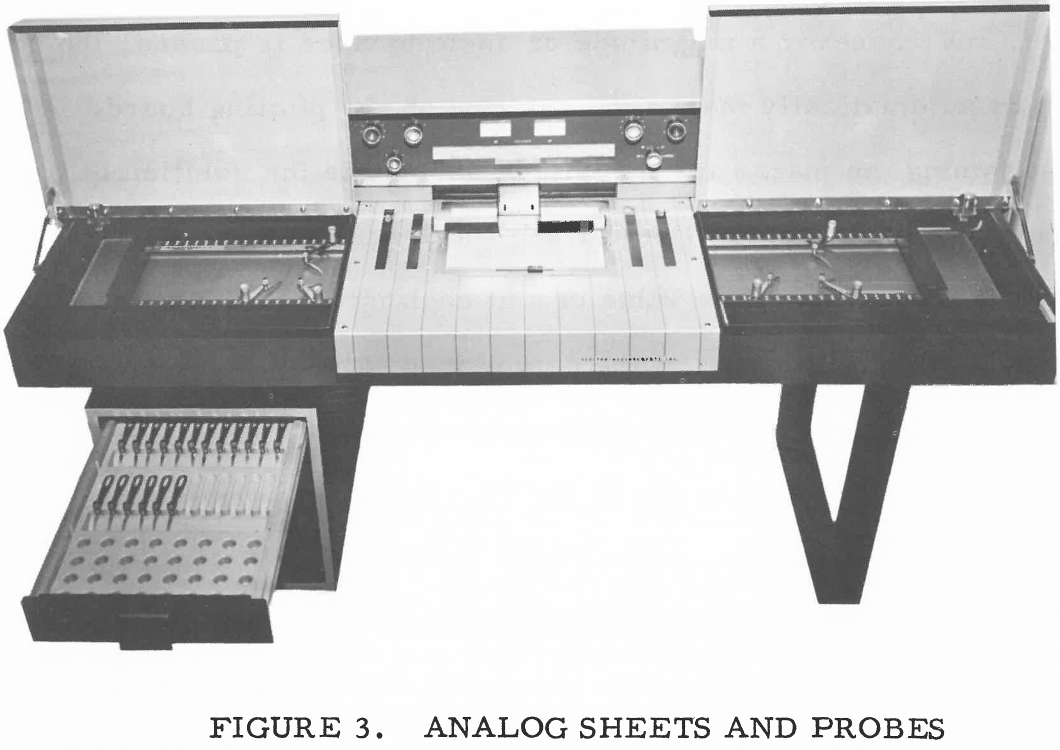


FIGURE 3. ANALOG SHEETS AND PROBES

the sign of K, the value of the exponent n_0 and the form of the general factor $(s - s_i)$ or $(1 - \frac{s}{s_i})$.

When the controls representing all the parameters in the equation are adjusted to any combination of values for which the two sides of the equation are equal, a pair of electrical null indicators, one for the magnitude and one for the angle, show the condition of equality. After the values of all given or assumed parameters have been set, the equation is solved simply by adjusting the value of the unknown until null balances are obtained.

For root finding or root locus plotting, the motion of the probe assembly is directly coupled to the stylus on the plotting board. For automatic plotting, the probe assembly is moved to scan values of s, and whenever a magnitude or angle balance is passed, the stylus automatically marks the position on the plotting board. A line joining the marks for magnitude balance is the solution of the magnitude equation; the line for angle balance is the solution of the angle equation. The value of s at each intersection of these lines is therefore a root of the complex equation.

To use the ESIAC for frequency response plotting, the position of the probe assembly in the coordinate representing $\angle s$ is fixed at 90°, and its motion in the $|s|$ direction is coupled to the plotting assembly in the horizontal direction. The vertical motion of the plotting stylus is coupled to the $|F|$ dial for gain plotting or to the $\angle F$ dial for phase plotting. When the circuit is balanced at various values of $|s|$, the stylus plots the log gain or phase angle versus log frequency.

Since the plotting stylus is directly coupled to the dials, it is not necessary to read the dials while making these plots. The stylus is simply moved about over the plotting paper and the automatic marking circuit plots the curves.

Logarithmic Coordinates

In the ESIAC, each parameter is represented in terms of its magnitude and angle. The magnitude scales are logarithmic, so that all values are represented with the same relative accuracy, as on a slide rule. The angle scales are linear. The complex plane on which values of s are plotted is therefore the log s plane rather than the s plane.

There are both electrical and mathematical advantages in using the log s plane. Electrically, the use of the log s plane makes possible the inclusion of both the $(s - s_i)$ and $(1 - \frac{s}{s_i})$ forms for the factors in Equation 1 and Equation 2. The two equations differ only in the value of K, but it is convenient to be able to use either K which may be specified in a problem.

Probably the most important advantage of the log s plane is that values of s over a wide range can be represented on the same plot. The ESIAC covers a range of four decades of s and eight decades of s_i's, thus including the effect of zeros and poles two decades beyond the ends of the plot. The use of the log s plane results in uniform relative accuracy over the whole four decades, and in compatibility between the scales for root loci and the scales for Bode plots.

In the log s plane, any complex quantity is repeated every multiple of 360° in the $\angle s$ direction. Therefore only one strip of 360° width is required for plotting all complex quantities. Any multiple of 360° can be added to or subtracted from the angle used to specify a complex quantity without changing the quantity. When analog voltages appear in the process of computation which are outside the 360° range of the $\angle F$ scale, a stepping switch automatically adds or subtracts multiples of 360° until the voltage is brought within the range.

Two types of graph paper, shown in Figure 4, are provided with the ESIAC for plotting values of s in the log s plane. The semi-logarithmic graph paper, ESIAC Form 46, has a logarithmic scale for $|s|$ and a linear scale for $\angle s$, corresponding directly to the scales on the ESIAC. These coordinates are the logarithmic transformation of the polar coordinates of the s plane. The transformation of the rectangular coordinates of the s plane, $\mathrm{Re}\,[s]$ and $\mathrm{Im}\,[s]$, produces the curvilinear coordinates of ESIAC Form 47. The four arms of the axes in the s plane, +, +j, - and -j, become four parallel horizontal lines in the log s plane.

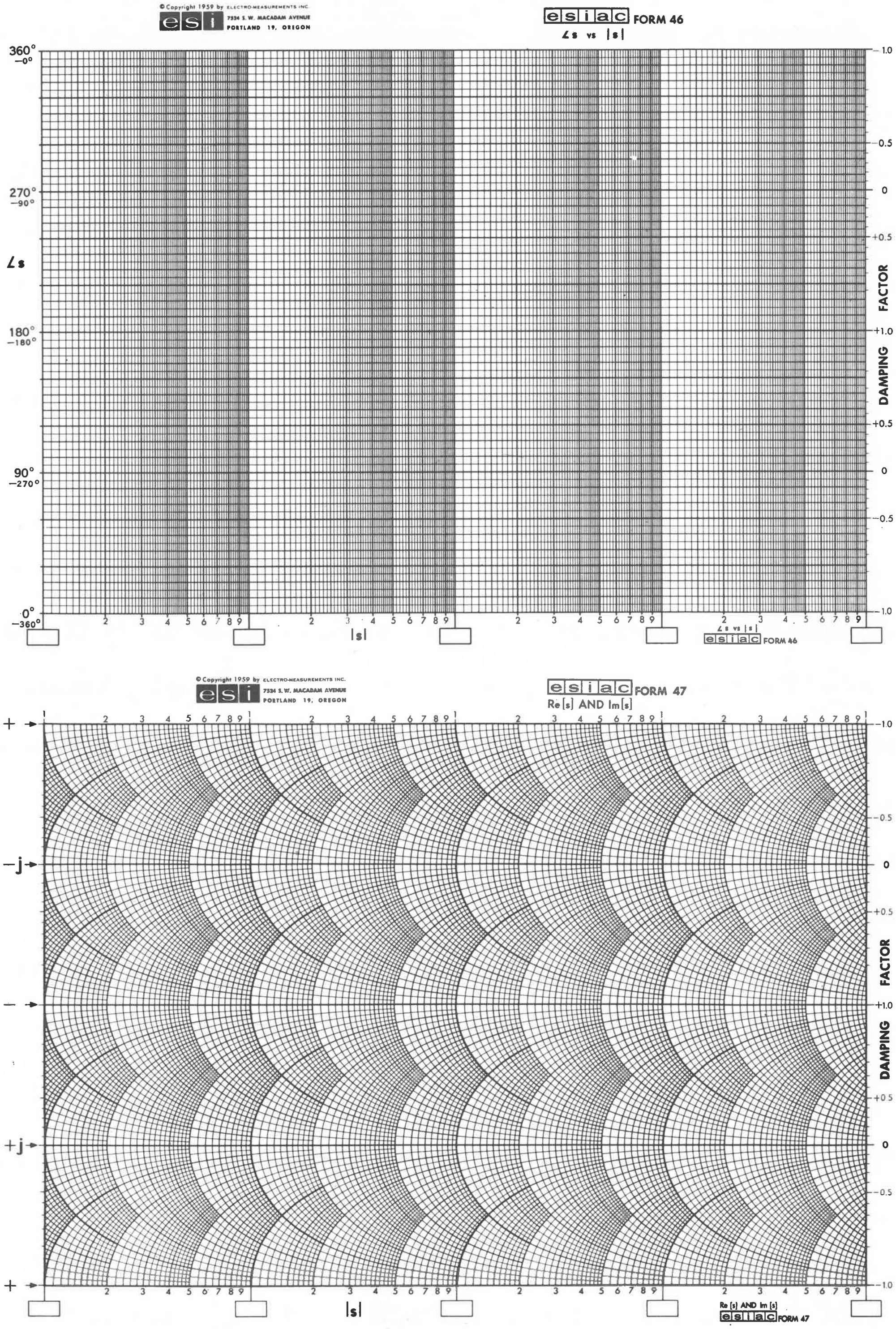


FIGURE 4. GRAPH PAPERS FOR THE LOG s PLANE

To read the value of s at any point on Form 47, it is necessary to follow the curves from the point to their intersections with the two adjacent parallel axes and read the coordinates there. Since all the axes have the same logarithmic scale, their values are marked only along the bottom and top edges of the graph paper.

In pole-zero diagrams or root locus plots, poles lying on the 180° axis represent critically damped responses. The damping of a response decreases as its poles move farther from 180° and reaches a state of continuous oscillation when the poles are at 90° and 270° respectively. Since the damping factor is the negative of the cosine of $\angle s$, contours of constant damping factor are parallel horizontal lines across the sheet. Both Forms 46 and 47 are provided with damping factor scales at their right margins as a convenience.

Frequency response plots on the ESIAC are simply plots of $|F|$ and $\angle F$ versus $|s|$ (angular frequency ω) or $\frac{|s|}{2\pi}$ (frequency f), when $\angle s = 90^\circ$. These appear as the familiar Bode plots of logarithmic gain or linear phase versus a logarithmic scale of frequency or angular frequency.

Typical ESIAC Applications

The ESIAC is applicable to a wide variety of problems encountered in control system analysis and design. It can be used on any problem which can be expressed in the form of equation

(1) or equation (2). Either F or s may be the unknown in the equation. The use of the ESIAC in a number of typical control system problems will be discussed briefly in the following sections. The discussions are not intended to be complete, but the general approach to various problems is indicated. Since the basic mathematical methods can all be found in standard control system textbooks, no references will be given.

Root Locus Plots

One of the most frequent applications of the ESIAC is in the construction of root locus plots. In general, a root locus plot is a curve of the locus of the roots of an algebraic equation as any parameter is varied. The term is most commonly applied to the determination of the poles of the closed-loop transfer function of a system by setting the denominator of the transfer function equal to zero and plotting the locus of the roots of this equation as the gain parameter K is varied. Since K affects only the magnitude balance of the equation, the locus is simply the solution of the angle equation, which is plotted directly on the ESIAC. An example of such a root locus plot for a moderately complicated control system is shown in Figure 5.

For a discussion of further details of such a problem, a much simpler example will be used. Its block diagrams and equations are given in Figure 6. Since two parameters K_a and K_t are to be chosen, two or more root locus plots will be made. The first,

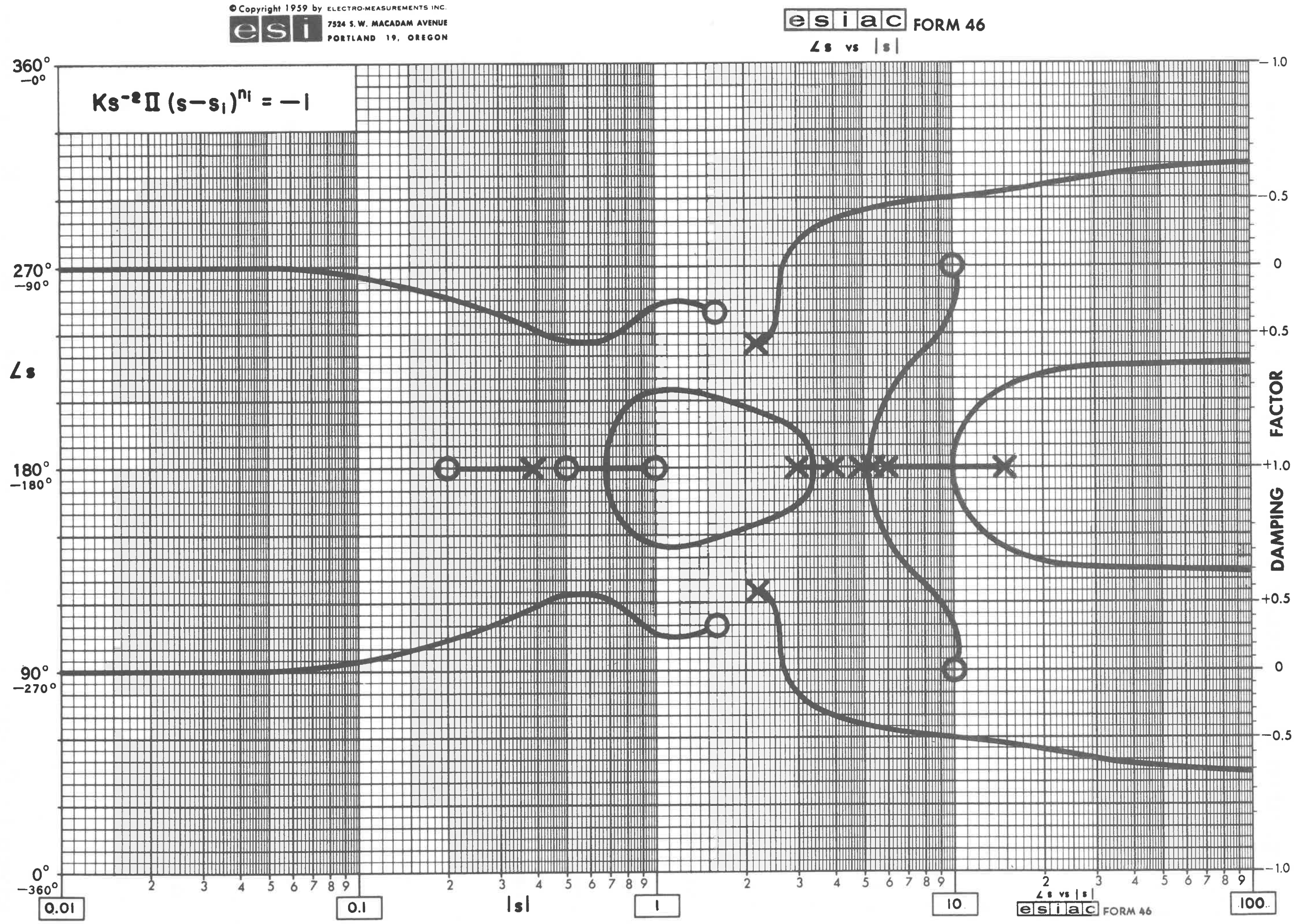


FIGURE 5. A SAMPLE ROOT LOCUS PLOT

shown in Figure 7, is used to locate the poles s_3, s_4 and s_5 of the portion of the system including the forward path and the unity feedback path. The variable parameter in this plot is K_a. A second plot can then be made for any chosen value of K_a to determine the poles s_6, s_7 and s_8 of the complete system as a function of K_t. For a choice of $K_a = 120$, the second plot is shown in Figure 8. Similar plots can be made if desired for other values of K_a by starting from their values of s_3, s_4 and s_5 in Figure 7. Note that the numerical values of s_3, s_4 and s_5 need not be read. Their graphical locations on Figure 7 are simply transferred directly to Figure 8. Assuming that K_t is to be chosen for the maximum damping factor of s_6 and s_7, their locations are determined by inspection in Figure 8. At the points indicated, the value $K_t = 0.18$ is read on the K scale of the ESIAC. The position of s_8 for this value of K_t is found to be as shown in the same figure. The numerical values of these roots can be read on Figure 8 as:

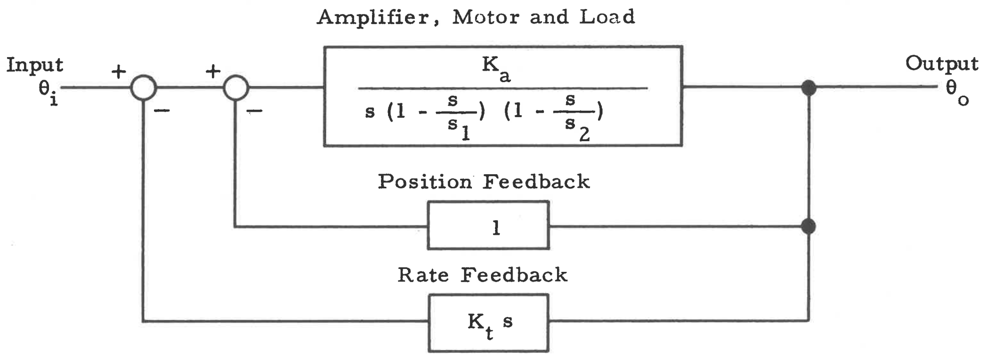
$$s_6 = -6 + j40$$

$$s_7 = -6 - j40$$

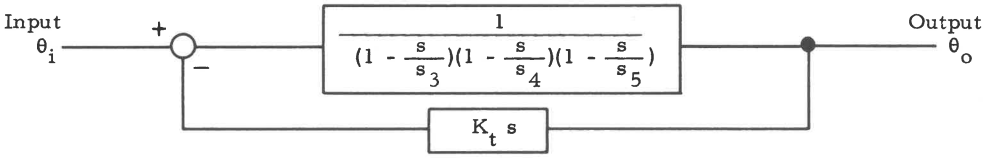
$$s_8 = -5.4$$

These are the final closed-loop poles for the system shown in Figure 6.

Block Diagram of System:


Amplifier, Motor and Load
Input
θ_i
+
−
+
−
$\frac{K_a}{s\,(1-\frac{s}{s_1})\,(1-\frac{s}{s_2})}$
Output
θ_o
Position Feedback
1
Rate Feedback
$K_t\,s$

First Reduction of Block Diagram:

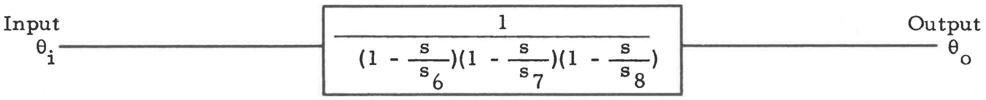

Input
θ_i
+
−
$\frac{1}{(1-\frac{s}{s_3})(1-\frac{s}{s_4})(1-\frac{s}{s_5})}$
Output
θ_o
$K_t\,s$

in which s_3, s_4 and s_5 are the roots of the equation

$$\frac{K_a}{s\,(1-\frac{s}{s_1})\,(1-\frac{s}{s_2})} = -1$$

For example, when $s_1 = -6$ and $s_2 = -12$, the roots are plotted as a function of K_a in Figure 7.

Second Reduction of Block Diagram:


Input
θ_i
$\frac{1}{(1-\frac{s}{s_6})(1-\frac{s}{s_7})(1-\frac{s}{s_8})}$
Output
θ_o

in which s_6, s_7 and s_8 are the roots of the equation

$$\frac{K_t\,s}{(1-\frac{s}{s_3})(1-\frac{s}{s_4})(1-\frac{s}{s_5})} = -1$$

For example when $K_a = 120$, the roots are plotted as a function of K_t in Figure 8.

For maximum damping, set $K_t = 0.18$. This places s_6 and s_7 at $-6 \pm j40$ for a damping factor of 0.15, and places s_8 at -5.4.

For these pole locations, the frequency response curves are plotted in Figures 9 and 10.

FIGURE 6. AN ILLUSTRATIVE PROBLEM FOR THE ESIAC

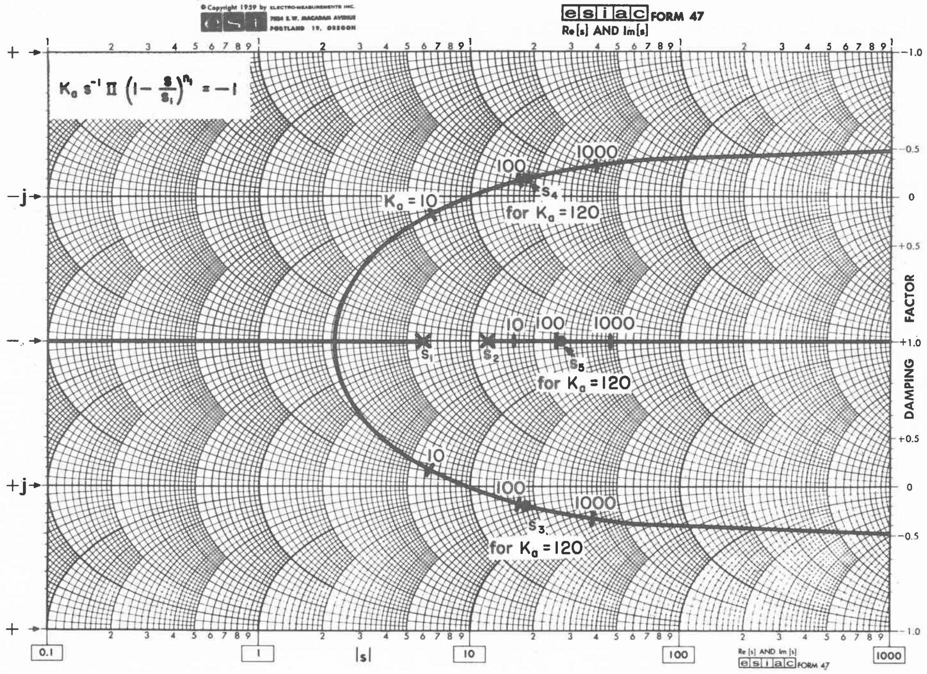


FIGURE 7. ROOT LOCUS PLOT FOR K_a

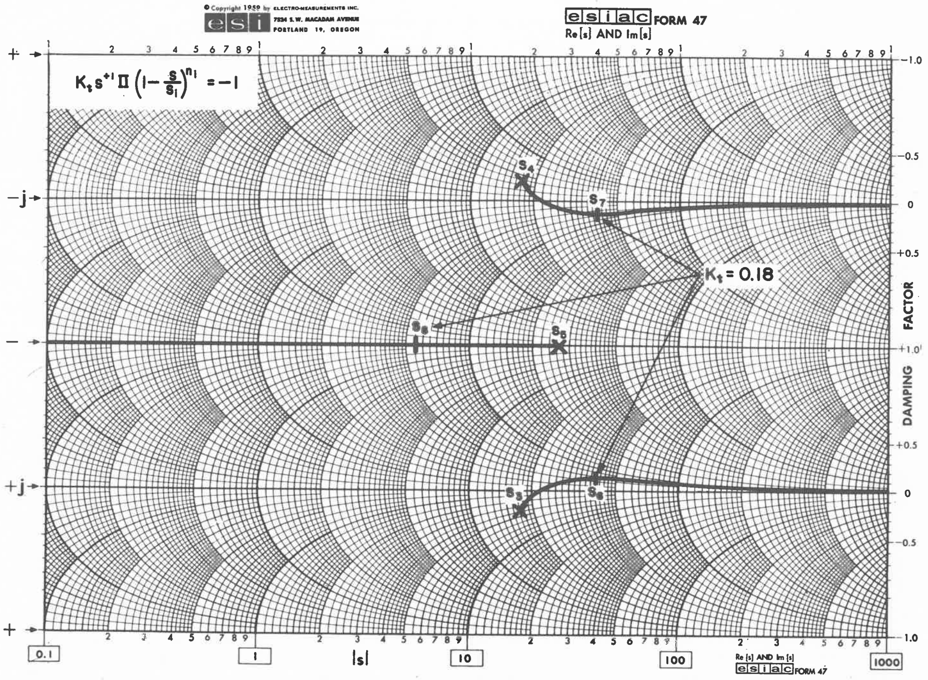


FIGURE 8. ROOT LOCUS PLOT FOR K_t

Transient Response by Partial Fraction Expansion

In making the inverse Laplace transformation of a response function by a partial fraction expansion, coefficients can be found by removing each pole factor and setting the variable to the value of the pole removed. The ESIAC performs these two operations simultaneously and gives the value of the coefficient or residue. The computer values can be used to find the complete partial fraction expansion even in the presence of high order poles.

In the system of Figure 6 assume that the input is a unit step $\theta_i = \frac{1}{s}$ The output of the system is then

$$\theta_o = \frac{9000}{(s - s_6)(s - s_7)(s - s_8)\, s}$$

or

$$\theta_o = \frac{A_6}{s - s_6} + \frac{A_7}{s - s_7} + \frac{A_8}{s - s_8} + \frac{A_o}{s}$$

By setting the factored function θ_o on the ESIAC, the residues at s_6, s_7 and s_8 can be read:

$A_6 = 0.07 \angle 80^\circ$ at $s_6 = -6 + j40$

$A_7 = 0.07 \angle -80^\circ$ at $s_7 = -6 - j40$

$A_8 = 1 \angle 180^\circ$ at $s_8 = -5.4$

By inspection of the function:

$A_o = 1$ at $s = 0$

Taking the inverse Laplace transform term by term, the output as a function of time is

$$\theta_o(t) = \mathcal{L}^{-1}\left[\theta_o(s)\right] = 0.14\, e^{-6t} \cos\left(80^\circ + 40t \frac{360^\circ}{2\pi}\right) - e^{-5.4t} + 1$$

Frequency Response Plots

If the frequency response of the system to a sinusoidal input is desired, it is easily plotted on the ESIAC. The final closed-loop gain and phase of the system of Figure 6 are simply the magnitude and angle respectively of the transfer function

$$F = \frac{\theta_o}{\theta_i} = \frac{1}{\left(1 - \frac{s}{s_6}\right)\left(1 - \frac{s}{s_7}\right)\left(1 - \frac{s}{s_8}\right)}$$

in which $s = j\omega$, and s_6, s_7 and s_8 are as shown in Figure 8.

In terms of the ESIAC controls, $s = j\omega$ means that $\angle s = 90^\circ$ and that $|s| = \omega$ (angular frequency) or $\frac{|s|}{2\pi} = f$ (frequency). Both $|s|$ and $\frac{|s|}{2\pi}$ are indicated directly on the ESIAC scale, and graph papers are provided for use with the ESIAC in plotting $|F|$ and $\angle F$ as a function of either ω or f. The plotting stylus in the ESIAC can be directly coupled to $|F|$ or $\angle F$ for plotting frequency response plots. The resulting curves as a function of f for this example are shown in Figures 9 and 10.

Factoring Polynomials

Problems are frequently encountered in which polynomials must be factored. Any polynomial can be expressed in either of the factored forms:

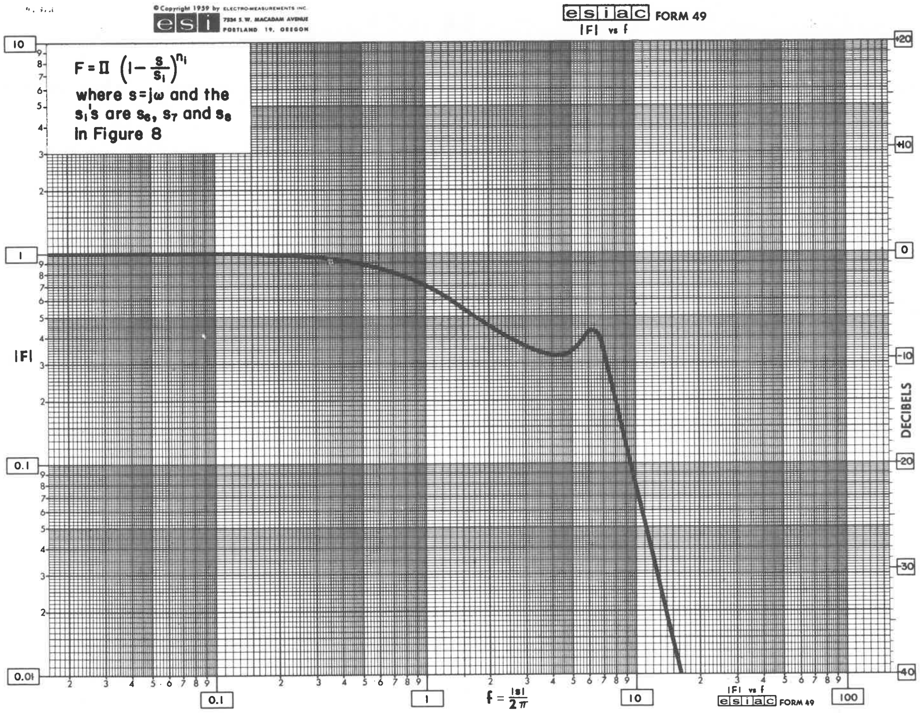


FIGURE 9. GAIN PLOT

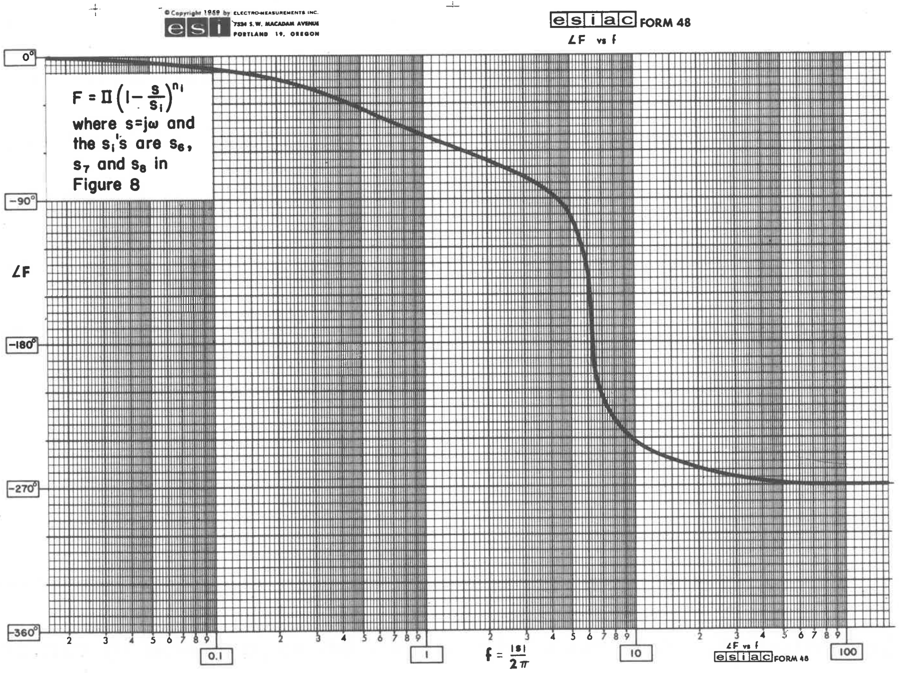


FIGURE 10. PHASE PLOT

$$a_o + a_1 s + a_2 s^2 + a_3 s^3 + \ldots + a_z s^z$$
$$= a_z \prod_{i=1}^{z} (s - s_i)$$
$$= a_o \prod_{i=1}^{z} (1 - \frac{s}{s_i})$$

in which the s_i's are the roots of the equation formed by setting the polynomial equal to zero.

This factoring can be accomplished on the ESIAC by a sequence of root locus plots, in which each plot is used to replace a group of terms in the polynomial by its factored equivalent. For example:

$$a_o + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4 + a_5 s^5 + a_6 s^6 \ldots$$
$$= a_o (1 - \frac{s}{s_1})(1 - \frac{s}{s_2})(1 - \frac{s}{s_3}) + a_4 s^4 + a_5 s^5 + a_6 s^6 + \ldots$$
$$= a_o (1 - \frac{s}{s_4})(1 - \frac{s}{s_5})(1 - \frac{s}{s_6})(1 - \frac{s}{s_7})(1 - \frac{s}{s_8}) + a_6 s^6 + \ldots$$
$$= \ldots$$

in which s_1, s_2 and s_3 are the roots of the equation

$$a_o + a_1 s + a_2 s^2 + a_3 s^3 = 0$$

s_4, s_5, s_6, s_7 and s_8 are the roots of the equation

$$a_o (1 - \frac{s}{s_1})(1 - \frac{s}{s_2})(1 - \frac{s}{s_3}) + a_4 s^4 + a_5 s^5 = 0$$

et cetera.

The equation for s_1 through s_3 is solved on the ESIAC by a root locus plot after rewriting it

$$\frac{a_o\left(1+\dfrac{a_1 s}{a_o}\right)}{a_2 s^2\left(1+\dfrac{a_3 s}{a_2}\right)} = -1$$

The equation for s_4 through s_8 is similarly solved after rewriting it

$$\frac{a_o\,(1-\dfrac{s}{s_1})(1-\dfrac{s}{s_2})(1-\dfrac{s}{s_3})}{a_4 s^4\left(1+\dfrac{a_5 s}{a_4}\right)} = -1$$

This procedure is repeated until the whole polynomial is factored.

Network Calculations

In network calculations, a variety of problems arise which require factoring or other operations involving root finding by locus tracing techniques. For example, when two factored transfer functions are to be added and the result expressed in factored form, the numerator factors may be obtained by a root locus plot. In this problem,

$$\frac{A(s)}{B(s)} + \frac{C(s)}{D(s)} = \frac{A(s)\,D(s) + B(s)\,C(s)}{B(s)\;D(s)}$$

The denominator is already factored but the factors of the numerator must be found by solving the equation

$$A(s)\,D(s) + B(s)\,C(s) = 0$$

This equation can then be expressed in the form

$$\frac{A(s)\,D(s)}{B(s)\,C(s)} = -1$$

and the roots found on the ESIAC.

Another similar problem is the determination of loading effect. When a network function Z(s) for infinite impedance output is known and a specified load resistance R(s) is to be connected, the transfer function of the cascaded system requires solution of the equation

$$1 + \frac{Z(s)}{R(s)} = 0$$

in which Z(s) is the driving point impedance of the unterminated network and R(s) is the terminating impedance.

Representing Dead Time on the ESIAC

In a system with dead time, such as a transport lag or a delay line, the expression e^{-Ts} appears. For calculations on the ESIAC, this expression is approximated by $\prod_i \left(1 - \frac{s}{s_i}\right)^{n_i}$, in which the zeros and poles s_i are located at a frequency slightly above the maximum signal frequency. This approach is the same of that of approximating a distributed delay line or filter by a network of lumped circuit elements.

Several approximations of this form may be made with different locations and orders of zeros and poles. An example using four zeros and four poles is given in Figure 11 .

Describing Functions for Non-Linear Systems

Many systems containing a non-linear element can be analyzed by the describing function method. In this method a Fourier analysis for the output of the non-linear element with a sinusoidal input is determined as a function of signal amplitude. It

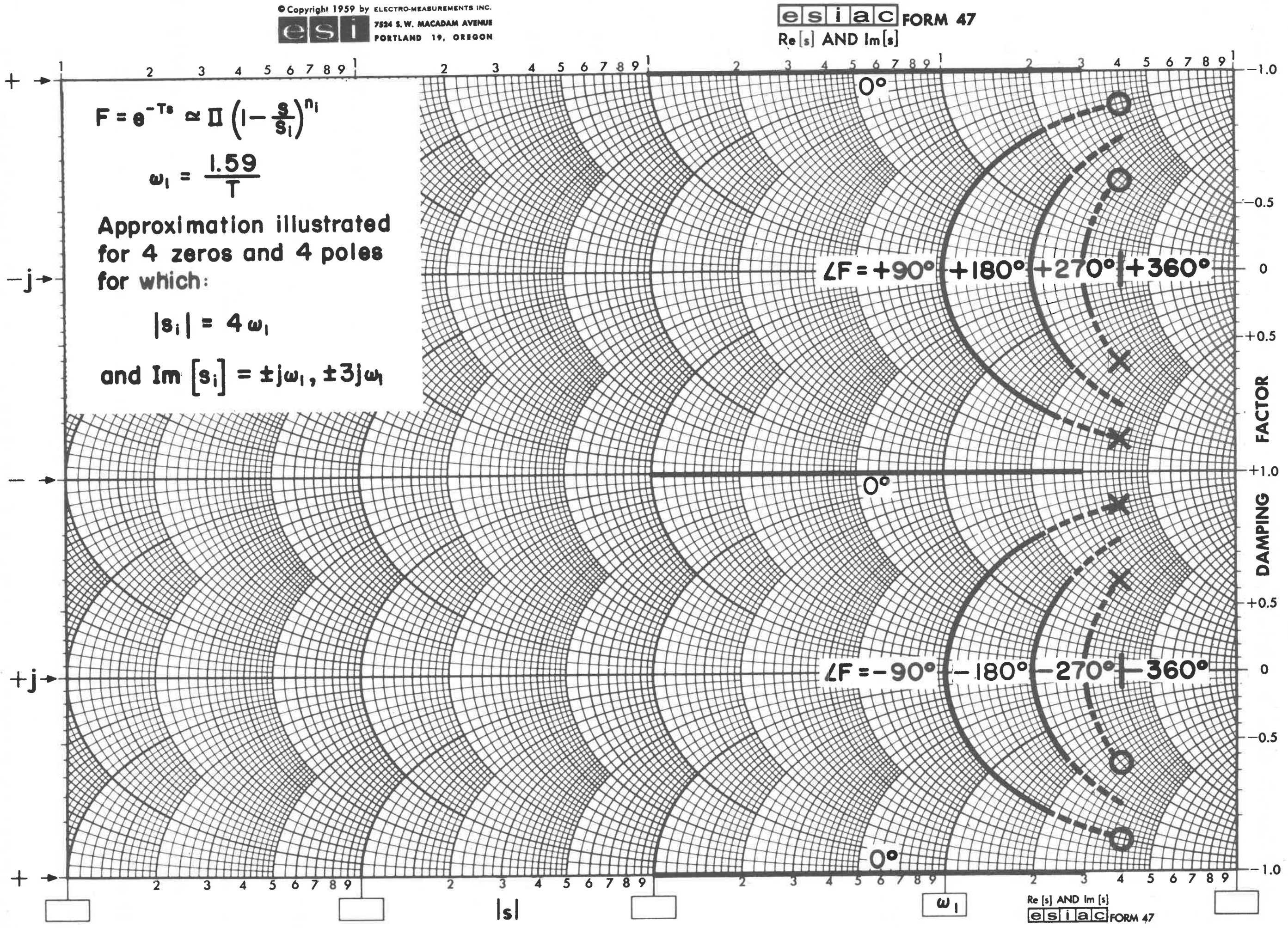


FIGURE 11. DEAD TIME APPROXIMATION

is assumed that only the fundamental component of the output of the non-linear element contributes to its input. This assumption is reasonable in many systems as a result of harmonic attenuation in the rest of the loop.

The gain and phase shift of the non-linear element at the fundamental frequency can be combined with the rest of the system in the same manner as for a linear element except that the transfer function for the non-linear element varies with amplitude. As a result, the analysis must be repeated for signals of various amplitude. The stability of a feedback loop containing such a non-linear element is determined by calculating first the sinusoidal transfer function for the linear part of the loop and then combining this result with the gain and phase shift of the non-linear element at various amplitudes. This analysis can be made using any of the usual techniques--Nyquist diagram, Nichols chart, Bode plot or root locus plot.

If the describing function contains a phase angle, the root locus method requires the plotting of loci for the linear portion of the transfer function not only for a 180° angle but also for other angles differing from 180° by the phase shift in the non-linear element. This added difficulty of construction has discouraged the use of root locus methods in such problems. With the ESIAC, however, the method is much more attractive since it is a very simple matter on the ESIAC to plot a family of loci.

Carrier Systems

In a carrier system, the closed-loop poles and system stability can be determined from the transfer functions before and after sinusoidal multipliers for modulators and demodulators. The transform of the modulator output is the input transform shifted in frequency by the carrier frequency. This is a sum of functions whose zeros can be determined on the ESIAC. When the poles and zeros of the carrier frequency components are superimposed, the operation is repeated for the demodulator.

Sampled Data Systems

The analysis of sampled data systems requires a transformation to the variable $z = e^{Ts}$. For calculations on the ESIAC, its variable s will be considered as z rather than s. In the z plane the frequency axis is the unit circle $|z| = 1.0$, not $j\omega$ as in the s plane. The frequency response is the magnitude and angle along this line of constant $|z|$. In the logarithmic representation of the ESIAC, this contour is a straight vertical line at $|z| = 1$.

Sampled data system functions are sometimes converted to functions of a new variable w defined by the relation

$$w = \frac{z - 1}{z + 1}$$

in which $z = e^{Ts}$ as in the previous method. These functions of w can be computed and plotted on the ESIAC in the same manner as functions of s.

Power-Density Spectra Problems

In power-density spectra methods, the poles and zeros of a power-density spectrum in the complex plane are given. There will be zeros and poles in all four quadrants of the complex plane. The spectrum can be measured along the frequency axis in the same manner as ordinary frequency response curves.

If a given input spectrum is passed through a system or a filter, the output spectrum is obtained by superimposing the pole-zero pattern of the transfer function and its mirror image in the frequency axis on the pole-zero pattern of the spectrum. For the cross-power spectrum between the system output and another signal, the pole-zero pattern of the complex conjugate of the transfer function is added to the pole-zero pattern of the cross-power spectrum between the system input and the other signal. The output cross-power spectrum will have the phase as measured along the frequency axis.

When the spectra of signal and equivalent noise at a system input are given, together with the transfer functions of an unalterable system components, the open-loop transfer function of the optimum control system can be determined by the Wiener-Hopf method.

It is possible to determine the open-loop and closed-loop transfer functions of an operating process by measuring the spectra at the input and the output of the controller for two different controller gain settings. The transfer functions can then be obtained by making root locus plots from this data.

All of the above power-density spectra methods involve algebraic operations which are conveniently performed on the ESIAC.

Conclusion

The ESIAC algebraic computer has been presented as a new tool for use in control system analysis and design. The ESIAC serves the designer as a "two-dimensional slide rule" for functions of a complex variable and their pole-zero plots. The ease with which it locates roots and plots graphs makes possible rapid comparison of various proposed stabilizing techniques and other design changes in terms of their algebraic transfer functions. The ESIAC is particularly useful in a sequence of calculations where each solution gives the designer further insight into the nature of the problem and thus influences the direction which the investigation should take.

It is hoped that this presentation of the features of the ESIAC and some of its applications will encourage greater use of these techniques, and perhaps even stimulate the development of better techniques than are now known.